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Introducing Feedback Control to Middle and High School STEM Students, Part 2: Control System Math ⋆

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Abstract: This paper presents some insights on introducing control concepts to middle and high school STEM students. It summarizes the author's experience in introducing control to such students at multiple workshops preceding control conferences. In Part 1 (Abramovitch (2019)), we share here the ideas behind the success of these talks. In Part 2 (this paper), we show how to discuss control systems math to students who have not had calculus or differential equations.

Keywords: Teaching, education, second-order systems, closed loops, computers.

1. INTRODUCTION TO PART 2

Part 1 of this paper (Abramovitch (2019)) discussed the need for a more complete public knowledge of the concepts behind, the ubiquity of, and the methods used in feedback control. It was focused on how to make these relevant to middle and high school students interested in Science, Technology, Engineering, and Math (STEM) subjects. One of the greatest challenges with such a task is that these students have almost universally not learned about differential equations, linear algebra, and transform theory. Only the upper level high school students are likely to have had a physics class or integral or differential calculus. This makes presenting the math that underlies control systems hard to explain. However, we do have some avenues still available to us.

What we can count on is that they have had some algebra, that they have seen geometric proofs (an introduction to theorem-proof methods), that they are familiar with finding the roots of a second order polynomial with the quadratic equation, and they are familiar with localizing roots of higher order polynomials.

We have found that students respond well to being told the following:

- To understand what the systems we want to control are doing and how our control will affect them, we need to model the systems using science (to derive models) and math (to put the models in a mathematically form that helps us predict what will happen next).
- It takes about 3 years of math from where the average student in the room is at that point. (It will be a bit more for the middle school students, a bit less for the upper level high school students.)
- We cannot hope to teach them the math they need in the time we are with them, but we can teach them what the math tells us and how it helps us.

The students respond well to this sequence. We can follow up by describing the kind of math that they will be learning. They will learn differential and integral calculus that will tell them the beginning of how things move. The integrals will allow them to do transforms (Laplace, Fourier, etc.) which have the convenient property that they turn differential equations into algebra. Once we get to algebra (which they know), their old friend of finding the roots, allows them to understand the system behavior. Now we have tied something they know to something we need to know to understand control systems.

2. NEWTON AND THE RIGID BODY

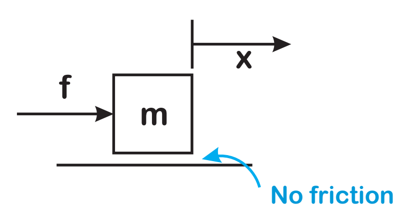

Fig. 1. A rigid mass on a frictionless plane is the starting point for understanding Newton's Law.

Most students in this age group will have seen at some level, Newton's Second Law: $f = ma$. We can tie this to a simple mass on a frictionless plane, as shown in Figure 1. Now, we remind them that acceleration is the second derivative of position. Some of the high school students have had differentiation, but most can understand that velocity is the change of position over time, and acceleration is the change in velocity over time. Thus we can rewrite $f = ma$ for them as:

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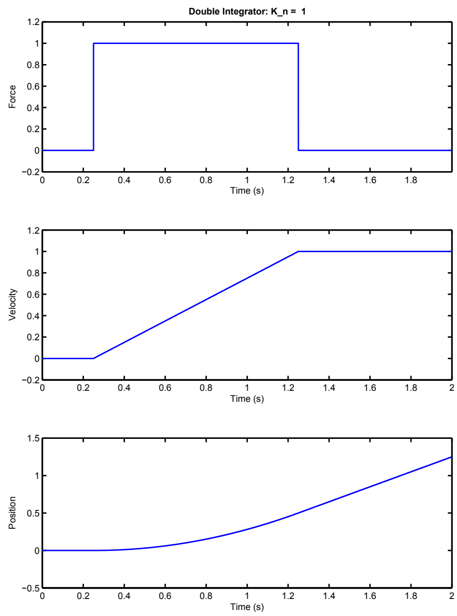

Fig. 2. Response of double integrator to force applied over finite time.

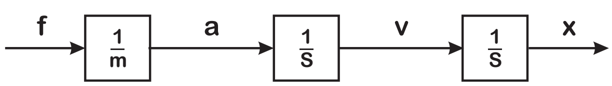

Fig. 3. The block diagram of the double integrator achieved when we ignore the initial conditions and use Laplace Transforms.

$$a = \ddot{x} = \frac{1}{m} f \longrightarrow v = \int_0^t a dt + v_0 \longrightarrow$$

$$x = \int_0^t v dt + x_0 = \int_0^t \left(\int_0^t a d\tau + v_0 \right) dt + x_0. \quad (1)$$

Again, they don't have to know how to do integrals, but we have led them to the idea that these matter and that they allow us to describe the physical system of Figure 1. Equation 1 has two integrals and so this system is called a double integrator, and it shows up a lot. One can imagine an air hockey puck on an infinite table. In each case, if one taps the puck, it has a constant velocity. However, if we push it with a constant force, it accelerates as long as the force is being applied. This is displayed in Figure 2. The top plot is the applied force, the middle plot is the velocity of the block, and the bottom plot is the position. We will use these types of plots to explain the behavior of different systems under control both open loop and closed-loop.

Here is where we need them to trust us. We tell them that there are a set of special integrals (Bracewell (1978)) which allow us to transform equations such as Equation 1 into algebra. Thus,

$$x = \int_0^t \int_0^t (a) dt dt \quad \longleftrightarrow \quad X(s) = \frac{1}{ms^2} F(s)$$

Time Domain (math) Transform Domain (2)

When we are done, from the transformed math we know that this system is not stable. That is, if we push the block on the frictionless plane, it will go on forever. Once we stop pushing, the velocity remains constant (middle plot), but the position keeps increasing. That being said, it is easy to control. That is, if we apply the same amount of force in the opposite direction for the same amount of time, the block stops. Finally, a lot of systems look like this (assuming we do not look too closely). Even without imagination, this shows up in most spacecraft control problems, since there is no air to generate friction and no spring force of gravity.

The above discussion has shown how we can lead the students along a path that is supported by a lot of knowledge they already have to the concept of a dynamic system and its stability. Not only have we discussed it qualitatively, but we have tied it into the physics of the problem (using an equation that they almost certainly know), and we have discussed some of the useful tools (transforms) that we use to understand the problem. Finally, they have seen an example of this behavior plotted out. With this, they are ready for the next step: adding feedback into this problem.

3. ADDING FEEDBACK TO DOUBLE INTEGRATOR

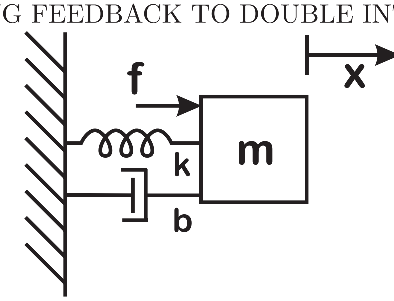

Fig. 4. Adding a spring and a damper to our original mass block.

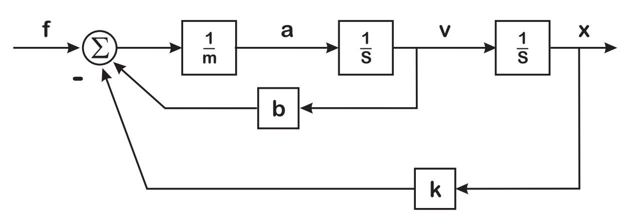

Fig. 5. The block diagram of the double integrator with velocity and position feedback.

One of the great pedagogical things about laying out the double integrator as we have done above is that now we can transform it to a spring-mass-damper system by adding a position feedback (k) and a velocity feedback (b). We are in a position to describe these not only in the picture of Figure 4, but in the block diagram of Figure 5. Equation 1 gets modified to be:

$$m\ddot{x} = f - b\dot{x} - kx \quad \longleftrightarrow \quad \frac{X(s)}{F(s)} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}},$$

Time Domain (math) Transform Domain (3)

where we see these feedback terms showing up explicitly in the time domain equation and this is transformed on

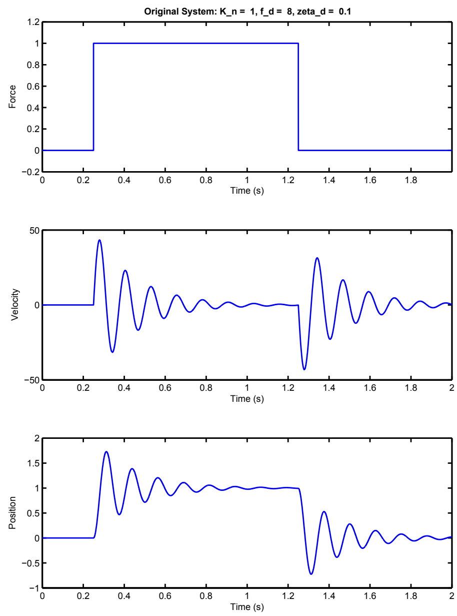

Fig. 6. Response of double integrator with velocity and position feedback to force applied over finite time.

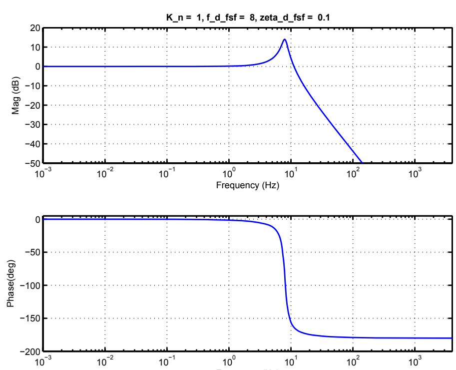

Fig. 7. We can even show them a Bode plot; not by telling them all the math, but by explaining to them what it means to us.

the right into a relationship which we can introduce to them as a transfer function. On the transfer function side, they see that if we set k and b to 0 we are back at our double integrator case. Furthermore, we can relate the second order relationships of k and b and m to those of an oscillatory system by matching coefficients in Equation 4 which relates the spring and damper parameters to that of a simple resonance:

$$\frac{X(s)}{F(s)} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{K\omega_d^2}{s^2 + 2\zeta_d\omega_d s + \omega_d^2}, \quad (4)$$

where

$$\sqrt{\frac{k}{m}} = \omega_d = 2\pi f_d \Longleftrightarrow f_d = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \quad (5)$$

and

$$\frac{b}{m} = 2\zeta_d\omega_d \Longleftrightarrow \zeta_d = \frac{b}{2\sqrt{km}}. \quad (6)$$

Now, they have almost certainly seen that when they try to solve for the roots of a quadratic polynomial and the number under the radical is greater than 0, the roots are real, but if the quantity is less than 0, things get weird. Some of them will know that this produces complex numbers. We can use this to introduce the idea of a Bode plot with Figure 7. Many STEM students will have seen something like this when they are looking at specifications for things like headphones (or ear buds). They might not understand what they saw, but it's straightforward enough to explain to them that the top plot would be like one they might have seen, except that they'd never want to buy ear buds if the response showed that resonance. They'd be looking for something flat. In buying ear buds, they would never see the lower plot, which is the phase – the relative angle of output to the input. Again, this is a teachable moment in that we can explain to them that worrying about this lower plot is one of the main differences between folks who work in signal processing and folks who work in feedback systems. We can come back to it later, but we can tell them that the plot relates the response and helps us understand what is going on even without a computer. We can tell them that from a plot like this, we can tell that the roots of that denominator polynomial are complex, and that if the system gets hit with something like a step in force, it will oscillate (ring) back and forth before settling down.

Now, we have gone, in very straightforward steps, from a double integrator system to one with feedback from the outputs of both integrators. We have shown them that using some math they don't have (differential equations and transform theory) and some math that they do have (algebra and root finding) that they can understand or "model" the behavior of this very physical system. Equations 5 and 6 tell us about the behavior:

- k/m tells us how fast it rings.
- b/m (in relationship to k) tells us how long it rings
- Making k bigger means the spring is stiffer, which results in higher frequency ringing.
- Making b bigger relative to k causes the ringing to damp out faster.
- Because denominator polynomial is 2^{nd} order, we can get roots with quadratic equation.
- Any polynomials that are more than 2^{nd} order are a lot harder.

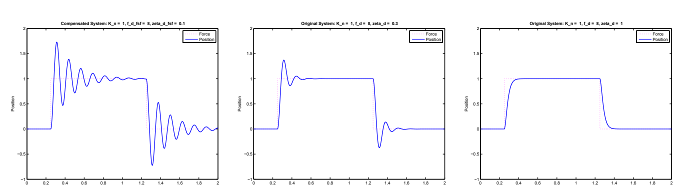

Fig. 8. Double integrator with spring and damper feedback. Resonant frequency (ω_d) at 8 Hz. We show the effect of changing the damping (ζ_d) from 0.1 on the far left, to 0.3 in the center, to 1 on the far right.

To illustrate the changes we can make by changing k/m and/or b/m we can show some relatively straightforward plots in Figure 8. Since we have described the damping and the oscillatory frequency as functions of b, k, and m, we can show how changing their relationship can change the damping and dramatically change the behavior of the system. We have introduced these as properties of the physical system itself. We are trying to understand/model the behavior with these equations and plots to gain insight.

This structure has set them up for the next step: introducing our own augmentation to nature's parameters.

4. INTRODUCING HUMAN AUGMENTATION OF NATURE'S FEEDBACK

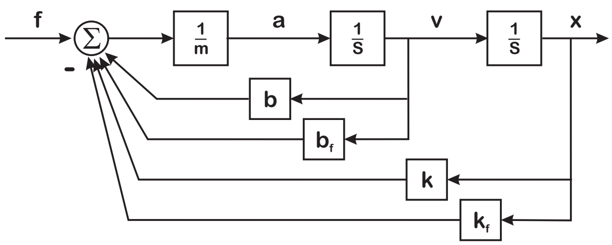

Fig. 9. Adding our own feedback to the spring-mass-damper system.

At this point, we have shown them the effects of nature's feedback parameters on the behavior of our simple system, and it is natural to ask, "What if nature's k and b are lame? Can we compensate?" The answer is – of course – yes (or we would all need to find new jobs), but we can use this model to show how we introduce augmented feedback into the system as shown in Figure 9. We can describe this as adding our own signals to feed back the output of each energy collector (which we call a state) back into the input of the system. If we do it right, we are – in the words of Shrek – compensating for something. We can analyze this to pick our parameters by looking at modifications of Equations 4–6.

$$\frac{X(s)}{F(s)} = \frac{\frac{1}{m}}{s^2 + \frac{b+b_f}{m}s + \frac{k+k_f}{m}} = \frac{K\omega_d^2}{s^2 + 2\zeta_{df}\omega_{df}s + \omega_{df}^2}, \quad (7)$$

where

$$f_{df} = \frac{1}{2\pi}\sqrt{\frac{k+k_f}{m}} \text{ and } \zeta_{df} = \frac{b+bf}{2\sqrt{km}}. \quad (8)$$

We have shown the students in a very straightforward way that we can augment nature's feedback with our own. We can also tell them that this is called "full state feedback", which is the 800 pound gorilla of control. (This gives one a chance to explain the 800 pound gorilla joke from our parents' generation.)

At this point, we can show them very simply that by cleverly choosing our feedback parameters, as shown in Figure 10, we can dramatically improve the system's behavior. By adding a little bit of damping (on the left) the system rings but eventually settles down. A bit more damping (center) and the system settles without ringing. If we maintain this new damping and artificially increase the stiffness of the spring, we get a higher frequency which means that the system responds even more quickly. This is analogous to cars that can adjust their suspensions to the driving conditions.

The next level is to explain what happens when we get the feedback parameters wrong. This can be simply illustrated by showing cases where the damping gets set to 0 (on the left of Figure 11) or even negative (on the right of Figure

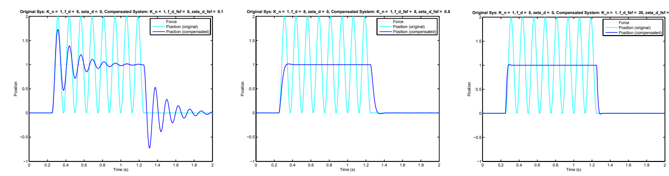

Fig. 10. Our spring mass damper with $\zeta_d = 0$ and $f_d = 8Hz$. The cyan curve shows this response, which rings without stopping. The blue curve shows the response of the system to the same input, when we humans have augmented nature's feedback. We then use our augmented feedback to change ζ_d to 0.1 (far left), ζ_d to 0.8 (center), and even change the frequency, f_d to 20 Hz.

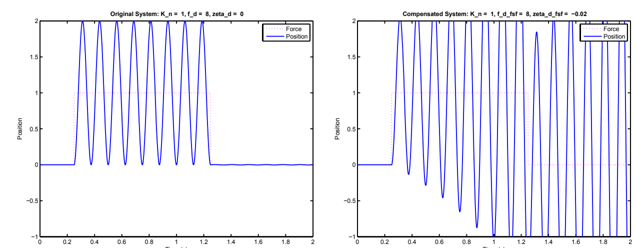

Fig. 11. If we accidentally set $b_f = -b$ (left plot) we zero out the damping of the original system and now it rings forever. If we accidentally set set $b_f = -1.02b$ (right plot) we have negative damping which gives us positive feedback, and this is bad.

11). The negative damping results in positive feedback and by then, they should have been told that when negative feedback becomes positive feedback, bad things happen – as illustrated by the ever increasing oscillations. Again, this can be tied back to physical examples such as a car swerving erratically because a driver under the influence is compensating for the weaving far too slowly.

Looking back at what we have done: we have introduced a simple system, shown how nature's feedback affects its behavior, added in our own feedback to improve the behavior, and shown the pitfalls of making a mistake. We have shown them modeling, feedback, full state feedback to augment nature's feedback, stability, and instability in a very simple progression. What is left to do? We can shown them what happens when we cannot measure everything and we can show them how we use math to help us deal with systems that are bigger than second order.

5. FEEDBACK CONTROL WITH FEWER MEASURED OUTPUTS

The curves of Figures 13 and 14 show us that when we only feed back information from position in this problem, something goes horribly wrong. How do we explain this to the students? We have to tell them about the math without expecting them to know how to do it themselves. We tell them instead, what it tells us. Here is where the transforms and Bode plots that we introduced before (knowing full well that they would not at this point be able to do the math themselves) tell us things. Since they have seen feedback loops in our prior discussion, we can show them

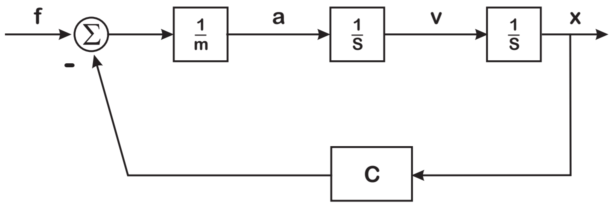

Fig. 12. The double integrator with feedback only from the position state.

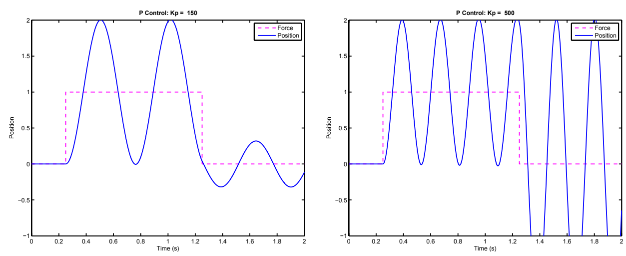

Fig. 13. Our double integrator system when we only feed back position. Here we have position feedback scaling (gains) of 150 (left) and 500 (right).

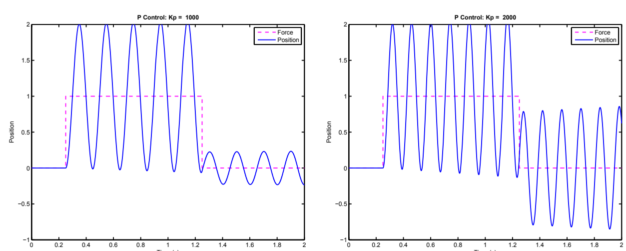

Fig. 14. Our double integrator system when we only feed back position. Here we have position feedback scaling (gains) of 1000 (left) and 2000 (right).

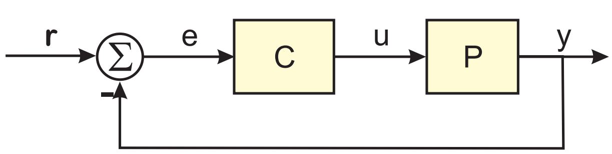

Fig. 15. Simplified abstract control loop. Transitioning from the complex block diagrams to this simple abstraction allows us to explain the uses of Equations 9 and 10.

the simplified model of Figure 15, which is an abstraction, something simplified so that we can get some insight from the math. We can tell them that in the transform space, we get ratios of polynomials from the models comprising Figure 15:

$$\frac{Y}{R} = \frac{PC}{1+PC} \quad \frac{E}{R} = \frac{1}{1+PC}$$
$$\frac{U}{R} = \frac{C}{1+PC} \quad \frac{Y}{U} = \frac{P}{1+PC}. \tag{9}$$

We can point out that all four of these have the same denominator and that this denominator governs a lot of behavior. We can tell them as they know about these that it's about finding the roots of the rational equation (where $1 + PC = 0$):

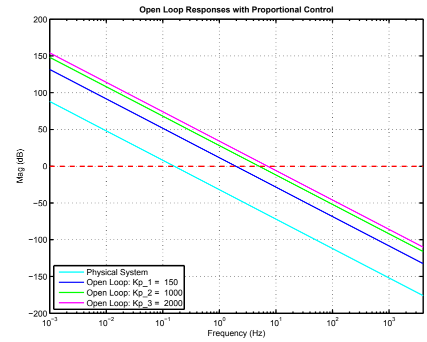

Fig. 16. Bode magnitude plots of the different levels of gain show that all we did was change the level of the sloped line, but did not alter the shape.

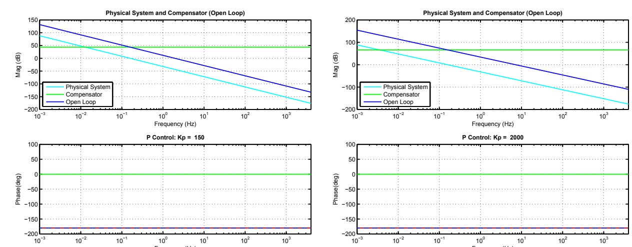

Fig. 17. Again, we use a Bode plot. On the left, our compensator gain is 150, on the right 2000. However, as neither has changed the phase, we eventually have a ringing of the system.

$$1 + PC = 1 + k_0 \frac{(s+b_1)(s+b_2)\cdots(s+b_m)}{(s+a_1)(s+a_2)\cdots(s+a_n)}. \tag{10}$$

Now, the critical points of PC are often very easy, but the critical points of $1 + PC$ are actually pretty hard – especially when one is in the 1940s/1950s and doesn't have a digital computer to help. We know from all the equations in Equation 9 that things get bad when $1 + PC = 0$. A long time ago, someone had the insight that this means:

$$PC = -1 \Leftrightarrow \|PC\| = 1 \;\&\; \angle PC = -180°. \tag{11}$$

This means that if we can check the magnitude and phase against each other, we can be careful that the magnitude should be less than 1 before the phase gets to $-180°$. This explains our fascination with Bode plots. The plots of Figure 17 show that no matter how big we make the gain, we have the problem that the phase is always at $-180°$ and so when the gain gets to 1 (which is 0 dB on this logarithmic plot), it will ring. It will ring at different frequencies depending upon our gain, but it will still ring.

6. FAKING MEASUREMENTS: ESTIMATION

In order to fix (or compensate for) our lack of velocity measurement, we know that we need extra information. This is pretty obvious to the students. While control theorists understand that we need some sort of estimate of velocity (or derivative information), this is hard to explain to students who have not yet computed a derivative. This author has found that the example of Figure 18 works spectacularly well, since almost every kid has thrown and caught a ball. The figure represents two images of balls

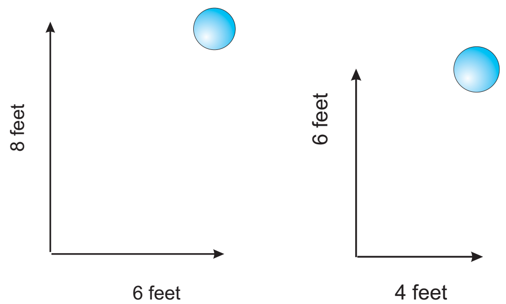

Fig. 18. Two images of a ball in the air with only position information for each of them. On their own, there is not enough information to catch a moving ball from these images. A time stamp is needed on each.

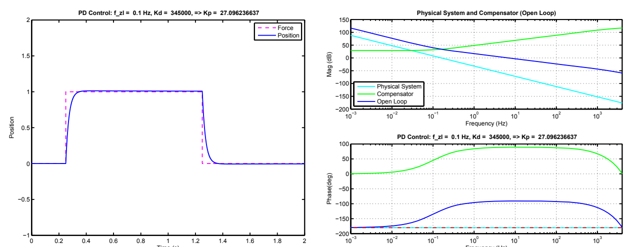

Fig. 19. Our double integrator system when we only feed back position information. Here we have used position and the derivative of position to make the feedback effective and stop the ringing. On the right, we can show them that the Bode plot helped us predict that this would happen and how to choose the relative and overall gains.

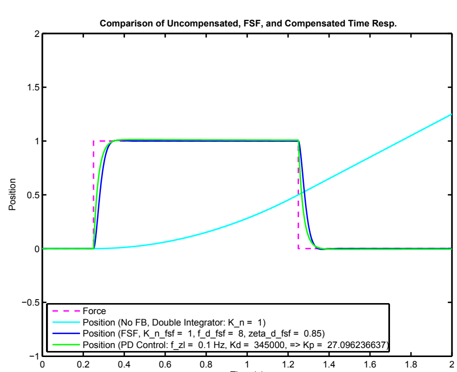

Fig. 20. Here, we repeat the plot on the left of Figure 19, because we have curves from full state feedback as well. Thus, we are showing that using some signals to estimate others can give us almost as good a result (sometimes) as when we can measure everything.

in the air with only position information. Having been through the prior material, we can all conclude that it is impossible from only that information for anyone to catch the ball. We then produce a soft ball and toss it to one of the students who almost always catches it. We then get a chance to explain the contradiction, that if the speaker is not a nonsensical liar, something else must have taken place.

That something can be explained as follows. Our eyes take repeated still images (position information). Our brains take the difference and add a time stamp. Our brains (meat computers) are estimating the velocity of the ball from changes in position over time. In fact, for many North American kids who have played baseball or softball, we can relate this to learning to play catch, how as their skill level improved, they could catch balls thrown much less accurately and with greater speed. This relates to their brain learning to produce better estimates over time. For kids from other parts of the world, similar analogies can be made with soccer (football), tennis, etc. They get the message: we need to teach our machine how to estimate the velocity.

Our system that fed back everything had no issues because it was getting velocity information. Can we somehow estimate velocity from our position measurement, and to do this, we need to differentiate, that is see how quickly our position is changing over time. We are now introducing proportional plus derivative control in doing this. In the transform space, we can point out that this makes the phase less negative. There is even a simple relationship for the new controller that we can show them:

$$C = k_f \frac{(s + b_f)}{(s + a_f)}, \quad 0 \leq b_f < a_f, \tag{12}$$

and point out two important things: that there is a set of algebra from our transform space that allows us to predict what math we need to do to estimate velocity and that this math is often very simple. We can also point out that a lot of very useful control systems have simple guts such as this.

When we do this, we get the response on the left side of Figure 19, which looks pretty good. On the right, we see that our Bode plot shows some differences, but the main thing that we show them is that the phase is well above -180° when the gain gets to 1 (0 dB). In fact, we can splurge a bit and show them Figure 20 which has the results of both the full state feedback and our PD feedback and we are showing them that with a good estimate, we can do almost as well as when we measure everything.

Again, it is worthwhile to look at the path we have taken with a set of bright students who have not yet studied calculus. We have shown them that we can do something with a limited set of signals to produce estimates of the signals we need to cause the system to behave the way we want. We have shown them that some math that they don't yet have leads to some math that they already know and that the math they already know tells us how to fix the problem, sometimes simply. We have attached physical meaning to the equations and examples that we use, so that even if they don't understand the details, they have a very strong sense that they get the basic idea.

7. SOME DEPTH ON ESTIMATION

While we have taken the students a long way in a short time with the previous discussion, there are some extensions and abstractions that are useful because they tie what we have been discussing to problems that they hear about every day. We start with the example of Figure 21. Here we have extended the simple physical example. We can then ask:

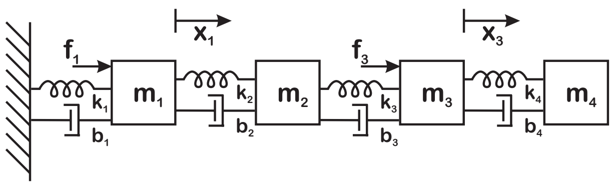

Fig. 21. Extending our spring mass damper system to one with a lot more springs, masses, and dampers.

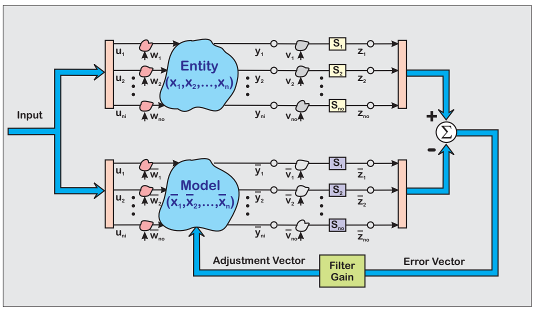

Fig. 22. A generalized view of model based filtering

1) What if we can't measure every "state"?
2) What if the physical system is more complicated than our model?

The trick is that since (2) is always true, then (1) is always true. This allows us to discuss really big, big systems with lots of stuff that we can't model exactly. Power grid, air traffic control, automated highways, systems biology, have many more things (states) than can be measured. However, these kids are comfortable with computers, with games, with simulations. This means that we can refer to Figure 22 as a general metaphor for how we handle these problems. The general process steps are:

- Build a simulation.
- Run simulation in parallel with real world.
- Compare simulation output to things we can measure in real world.
- Correct simulation with measurements from the real world.

In doing so, we get a lot of advantages, including that the inside of our simulation will have useful information we could not measure in the real world and that the student gets to tell their friends that they work with models.

8. CLOSING COMMENTS FOR MIDDLE AND HIGH SCHOOL STUDENTS

With such a whirlwind tour of control and system theory, the students might very well be energized, but not know where to go with the information. It is best to take a step back and give them an overview here, some thoughts about control systems and tech work in general:

- A lot of this discussion shows up in all technical work. Ideas about making measurements, using science to generate models, transforming those models into mathematics, using those math models to make predictions and improve design, and implementing things using computer programming are fairly universal..
- Computation has gone digital, not because digital gives better performance, but because digital gives cookie cutter, which gives miniaturization which gives more capability in small spaces everywhere. (They will recognize this immediately from their own experiences with smart devices.)
- To do useful things that touch the real world, you have to understand the real world and this usually takes modeling, and modeling takes math. (This answers the high school students inevitable questions in math class: "Are we ever going to use this?")
- Some folks do math for its own sake. Scientists and engineers tend to do math in order to enable them to do something practical. (This draws the distinction between different branches of STEM work in a relatable way.)

Again, at this point, we are trying to capture their imaginations for what they can do with all those math, science, and programming classes. This author always closes with the following set of comments about control systems and tech work in general:

- I'm not telling you this stuff is easy. It's not.
- Technical work takes work. There is a lot to learn, whether you want to be a biologist, a computer scientist, a chemical engineer, or a controls engineer.
- And every time you work on a new problem, you learn stuff you wish you had known on your last problem.
- What makes it fun?
  - A lifetime of learning.
  - Understanding the world.
  - Doing something about it.
  - Getting paid well in the process.

9. CONCLUSIONS FOR PART 2

This author's experience in giving such seminars to middle and high school students in a variety of pre-conference workshops has led them to believe that this pattern is highly successful and can be adopted by others. In Part 1 (Abramovitch (2019)) we presented the basic concepts that can be readily taught. In this part, we have tried to show how we can use some simple examples to tie Newton's Laws to control theory. We discuss how to explain what the math tells us without having to teach them the math itself. Furthermore, we discuss how some of that math allows us to turn these problems into some that they know, i.e. algebra. This allows us to display some algebra to explain what the math tells us about the behavior of control systems. These methods have worked extremely well for the author, and it is hoped that others can use them to get more middle and high school students interested in control engineering.

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